Dr. S. K. Yadav; email: sunil.phy30@gmail.com

In the last lecture we have discussed surface integral and how to evaluate it for the given problem. Here we discuss volume integral.

For a vector field $\vec{V} = i V_x + j V_y + k V_z$, and for volume element $d\tau = dx\,dy\,dz$, the Volume integral is given by $\int_v \vec{V} d\tau$

or $\int_v \vec{V} d\tau = i \int_v V_x d\tau + j \int_v V_y d\tau + k \int_v V_z d\tau$.

here $\int_v \vec{V} d\tau = \iiint \vec{V}\, dx\,dy\,dz$.

<u>Note</u>: In place of the vector function $\vec{V}$, ~~one can~~ we can also evaluate volume integral of a scalar function $f(x,y,z)$ (say).

<u>Q.1</u> Evaluate $\nabla \cdot \vec{V}$ over the unit cube, i.e. $0 \le x \le 1$, $0 \le y \le 1$, $0 \le z \le 1$. $\vec{V}$ is given by the vector function $i\,xz - zj + y z^2 k$.

Soln. We have to evaluate $\int_{\text{Unit Cube}} \nabla\cdot\vec{V}\, d\tau$

First, calculate $\nabla\cdot\vec{V} = \frac{\partial}{\partial x}(xz) + \frac{\partial}{\partial y}(-z) + \frac{\partial}{\partial z}(y^2 z)$

~~Here $\vec{V} = ixz - j z + k y$~~

$\vec{\nabla} \equiv i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$

$\vec{V} = ixz - jz + ky^2 z$

$\nabla\cdot\vec{V} = z + y^2$

$$\int \nabla\cdot\vec{V}\, d\tau = \int_{x=0}^{1}\int_{y=0}^{1}\int_{z=0}^{1} (z+y^2)\, dx\, dy\, dz$$

$$= \int_{y=0}^{1}\int_{z=0}^{1} (z+y^2)\, dy\, dz$$

$$= \int_{y=0}^{1} \left(\frac{1}{2}+y^2\right) dy$$

$$= \frac{1}{2} + \frac{1}{3}$$

$$\boxed{\int_{\text{Unit cube}} \nabla\cdot\vec{V}\, d\tau = \frac{5}{6}}$$

H.W. ① Evaluate $\int_{\text{Unit Cube}} \nabla\cdot\vec{V}\, d\tau$, where $\vec{V}$ is the vector field $ix^2y + jyz + ky^2z$. Integral is over unit cube $0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1$.

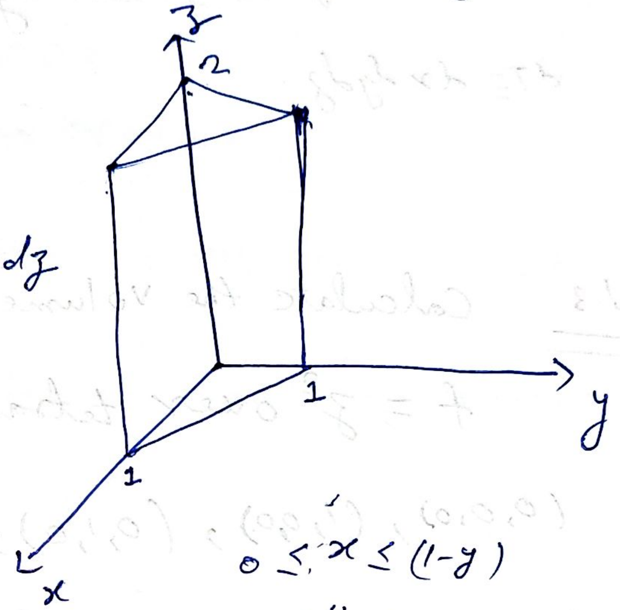
Q.2 Evaluate $\int_{v} \vec{V}\, d\tau$ over a prism defined by

$0 \le x \le (1-y)$, $0 \le y \le 1$, $0 \le z \le 2$ and vector function $\vec{V}$ is given by $4\hat{i} + 3y^2\hat{j}$. Here $d\tau = dx\,dy\,dz$

Soln.

We need to evaluate

$$\int_{v} \vec{V}\, d\tau = \int_{x=0}^{1-y}\int_{y=0}^{1}\int_{z=0}^{2} (4\hat{i} + 3y^2\hat{j})\,dx\,dy\,dz$$



One can evaluate the three integration in any order. Let us choose to do x integration first

$0 \le x \le (1-y)$

$0 \le y \le 1$

$0 \le z \le 2$

Figure shows the prism defined by x, y, z limits.

$$\int_{v} \vec{V}\, d\tau = \int_{y=0}^{1}\int_{z=0}^{2} \left[4x\hat{i} + 3y^2 x\hat{j}\right]_0^{1-y} dy\,dz = \int_{y=0}^{1}\int_{z=0}^{2} \left[4(1-y)\hat{i} + 3y^2(1-y)\hat{j}\right] dy\,dz$$

or
$$\int_{v} \vec{V}\, d\tau = \int_{z=0}^{2} \left[(4y - 2y^2)\hat{i} + 3\left(\frac{y^3}{3} - \frac{y^4}{4}\right)\hat{j}\right]_0^1 dz$$

$$= \int_{z=0}^{2} \left[2\hat{i} + \frac{1}{4}\hat{j}\right] dz = \left[2z\hat{i} + \frac{z}{4}\hat{j}\right]_0^2 = 4\hat{i} + \frac{1}{2}\hat{j}$$

~~or $\int_{v} \vec{V}\, d\tau = 2\hat{i} + \frac{1}{4}\hat{j}$~~

or
$$\boxed{\int_{v} \vec{V}\, d\tau = 4\hat{i} + \frac{1}{2}\hat{j}}$$

H.W. 2

Integrate $\int_{v} \vec{V} d\tau$ over the Prism defined by $0 \le x \le 1$, $0 \le y \le 3$, $0 \le z \le (1-x)$. and vector function $\vec{V}$ is given by $x^2\hat{i} + 2y\hat{j} + xz\hat{k}$. Here $d\tau \equiv dx\,dy\,dz$.

H.W. 3

Calculate the volume integral of the function $f = z^2$ over tetrahedron with corners at $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, $(0,0,1)$.

Hint: You have to evaluate $\int_{\text{tetrahedron}} f\, d\tau$. Note that here f is a scalar function. The process of integration will be same as shown in the solutions on the previous pages.